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FINITE INDENTATION OF AN ELASTIC MEMBRANE BY A SPHERICAL INDENTER

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ERRATA

Page	Line	Correction
iv	4	For "Contract" read "Contact" For "Indentor" read "Indenter"
1	22	For "is to the membrane r" read "to the membrane is r"
1	24	For "Solution" read "A solution ..."
4	8	For "with" read "With ..."
5	2	Should read: "equations of equilibrium [Eqs. (2.1a,b) below with $p = 0$], ..."
6	1	For "INDENTOR" read "INDENTER"
7	9	Should read: "The differential operator transforming as [Eqs. (2.6) and (2.7)]"
7	10	Comma after equation.
7	11	Should read: "Eq. (2.5b) can then be written"
9	2	For "(2.19a,b)" read "(2.14a)"
14	12	For "since" read "Since"
17	1	For "differential" read "derivative"
17	7	For "Eq. (3.22a)" read "(Eq. 3.21a)"
17	9	Should read "This, with the use of the left inequality (3.20), yields"
17	15	Should read "The total differential of the left-hand side of Eq. (3.12) yields an expression for the"
18	14	For "$C_o \neq C_{oL}$" read "$C_o = C_{oL}$"
21	8	For "yeilds" read "yields"; comma at end of line.
22	4	For "(3.38a)" read "(3.38)"
22	17	For "$P \leq P_\ell$" read "$P \leq P_L$" .
24	10	Read this line as "let $\sigma_L = \sigma_y = 10^4$ psi., and calculate $F_o = 1.44$, $P_L = 0.0704$ lbs, and $F_{oL} = 6.4$. Also $(\sigma_{r_o}/\sigma_L)^2 = (\sigma_{r_o}/\sigma_y)^2$ $= 7.2 \times 10^{-3}$;"
24	11	For "yeild" read "yield".
24	22	For "Limiting" read "The limiting..."

ABSTRACT

Rotationally symmetric stresses and deformations are considered for a prestressed elastic sheet of circular outer boundary loaded transversely by a centered indenter with a hemispherical tip. A nonlinear membrane solution is obtained for the portion of the sheet that is in frictionless contact with the rigid tip of the indenter. This solution and the solution that was previously obtained for a prestressed annular membrane [Ref. 1] are used to obtain the exact solution for stresses and deflections in the indented membrane. Simpler, limiting expressions for stresses and deflections are also obtained for sufficiently large and sufficiently small indenter loads.

Comparison of computed results from this analysis with the experimental results of Jansman, Field and Holmes [Ref. 2] on stretched mylar membranes shows good agreement except in the immediate neighborhood of the indenter. The theory shows that yielding and plastic deformation of the membrane is incipient for a relatively small value of the indenter load, and, for these experiments, this limiting value is less than the least value of indenter load for which data was reported.

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NOTATION

a	Outer radius of membrane.
b	Radius at the point of tangency.
c	Radius of the hemispherical head of the indenter.
C_o	Integration constant, Eqs. (1.7a,b).
$C_o(F_o)$	Function determined by Eq. (3.12).
E	Young's modulus.
f	$\equiv y^{1/2} \operatorname{ctn} \beta$.
F	$\equiv 2^{2/3} (\epsilon\rho)^{1/3} f$.
F_o	$\equiv \left(\frac{H_o}{Eh}\right)\left(\frac{4\pi Eha}{P}\right)^{2/3}$, combined loading parameter .
F_o^*	Value of F_o at which $C_o(F_o) = 0$.
F_{oL}	Value of F_o for $P = P_L$.
F_ϵ	$= F(\epsilon^2) = \left(\frac{2P}{\pi Eha}\right)^{1/3} \frac{c}{a}$.
$F_{\epsilon L}$	Value of F_ϵ for $P = P_L$.
h	Thickness of elastic sheet.
H_o	$\equiv h\sigma_{ro}$, Horizontal component of stress resultant at $r = a$.
P	Central indenter load.
P_L	$\equiv 4\pi Ehc\, s_L^2$, upper bound on P at elastic limit.
p	Transverse pressure on the elastic sheet under the indenter.
r	Radial coordinate.
$s(\beta)$	$\equiv s_r + s_\theta$.
s_r, s_θ, s, s_L	$\equiv (\sigma_r/E)$, (σ_θ/E), $[(\sigma_r+\sigma_\theta)/E]$, (σ_L/E) .
u	Horizontal displacement.
w	Transverse (vertical) displacement component.
w_o	$= -w(o)$; Central deflection.
y	$\equiv (r/a)^2$.

Notation (Continued)

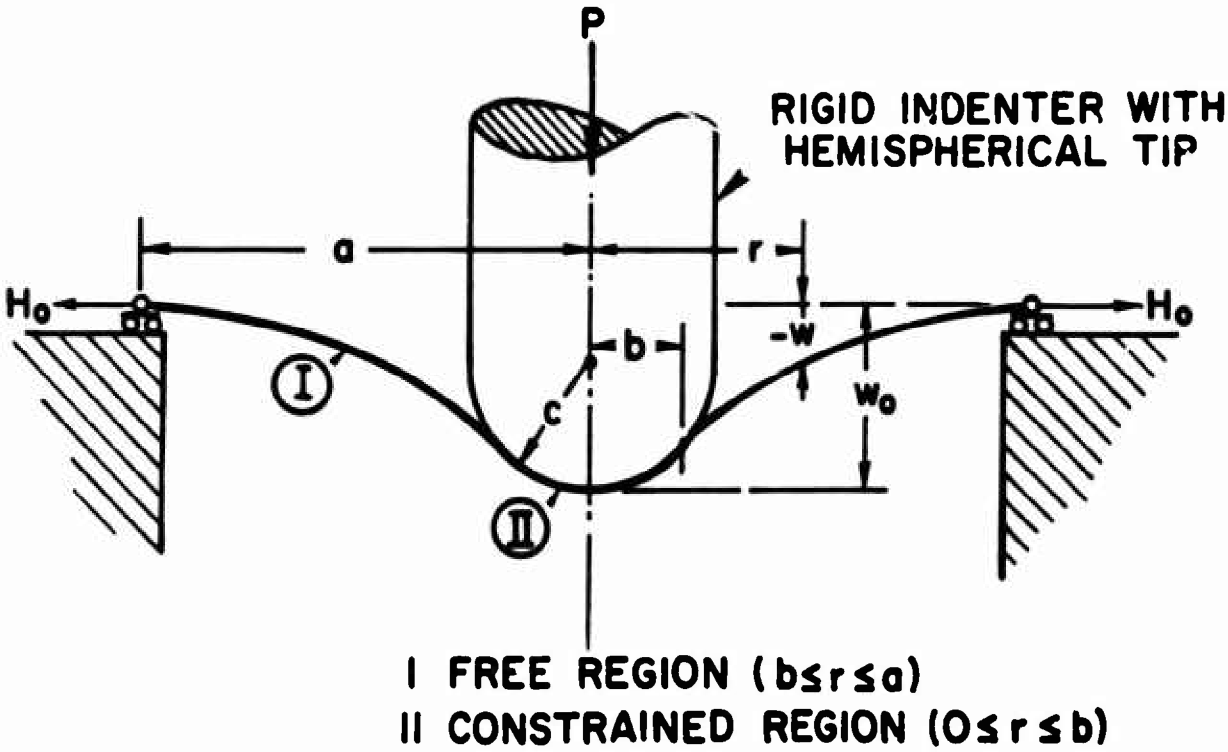
β	Angle of tangent rotation.
β_b	$\cong (b/c)$.
ϵ	$\equiv (b/a)$.
ϵ_r	Radial mid-surface strain.
ϵ_θ	Circumferential mid-surface strain.
ν	Poisson's ratio.
ρ	$\equiv \left(\frac{P}{2\pi Ehb}\right)$.
σ_r, σ_θ	Radial and circumferential stress components.
σ_{ro}	Applied prestress.
σ_L	Elastic proportional limit stress.

SECTION 1. INTRODUCTION

A theoretical analysis has been given by Nachbar [Ref. 1] for the finite, rotationally symmetric deformations and stresses of a prestressed, annular elastic membrane, or sheet, subjected to applied transverse loading only along the inner edge. The sheet is initially flat and is supported at the outer edge where stretching is also applied. An explicit solution was obtained for the case of transverse loading introduced through a rigid plug or disk which is attached centrally to the annular membrane. Experimental results for a similar problem have also been given by Jahsman, Field and Holmes [Ref. 2]. However, the experiments used a complete membrane and a rigid indenter with a hemispherical tip in order to apply transverse load. Thus, stresses and deformations of the membrane in the immediate neighborhood of the indenter tip are to be expected to be different from those predicted by the plug analysis.

In the present paper is considered the problem of finite, rotationally symmetric deformations and stresses in a prestressed circular flat sheet, of outer radius a, due to transverse loading at the center by a rigid indenter with a hemispherical tip of radius c. This problem will henceforth be referred to as the indenter problem. A small strain, elastic solution for the indenter problem is obtained. Figure 1 shows the indenter problem geometry and the nomenclature. The radial distance is to the membrane r, and $r = b$ denotes the point of tangency of the sheet with the indenter; b is a function both of load P and of H_o. Solution for the portion of the sheet $(0 \leq r \leq b)$ in frictionless contact with the indenter is obtained in Section 2. This portion will be called the constrained region. This solution and the results for the free region (annular portion of the membrane which is not in contact with the indenter) are used in Section 3 to obtain the complete solution to the indenter problem.

The results for the free annular region [Ref. 1] which are essential for the present paper are summarized below. These are valid for small strain and moderate rotation restrictions. The inner radius is b and the outer radius is a. Displacements normal and parallel to the initial



DEFORMED EQUILIBRIUM CONFIGURATION

Fig. 1. Deformation of the Elastic Sheet — Deformed Equilibrium Configuration.

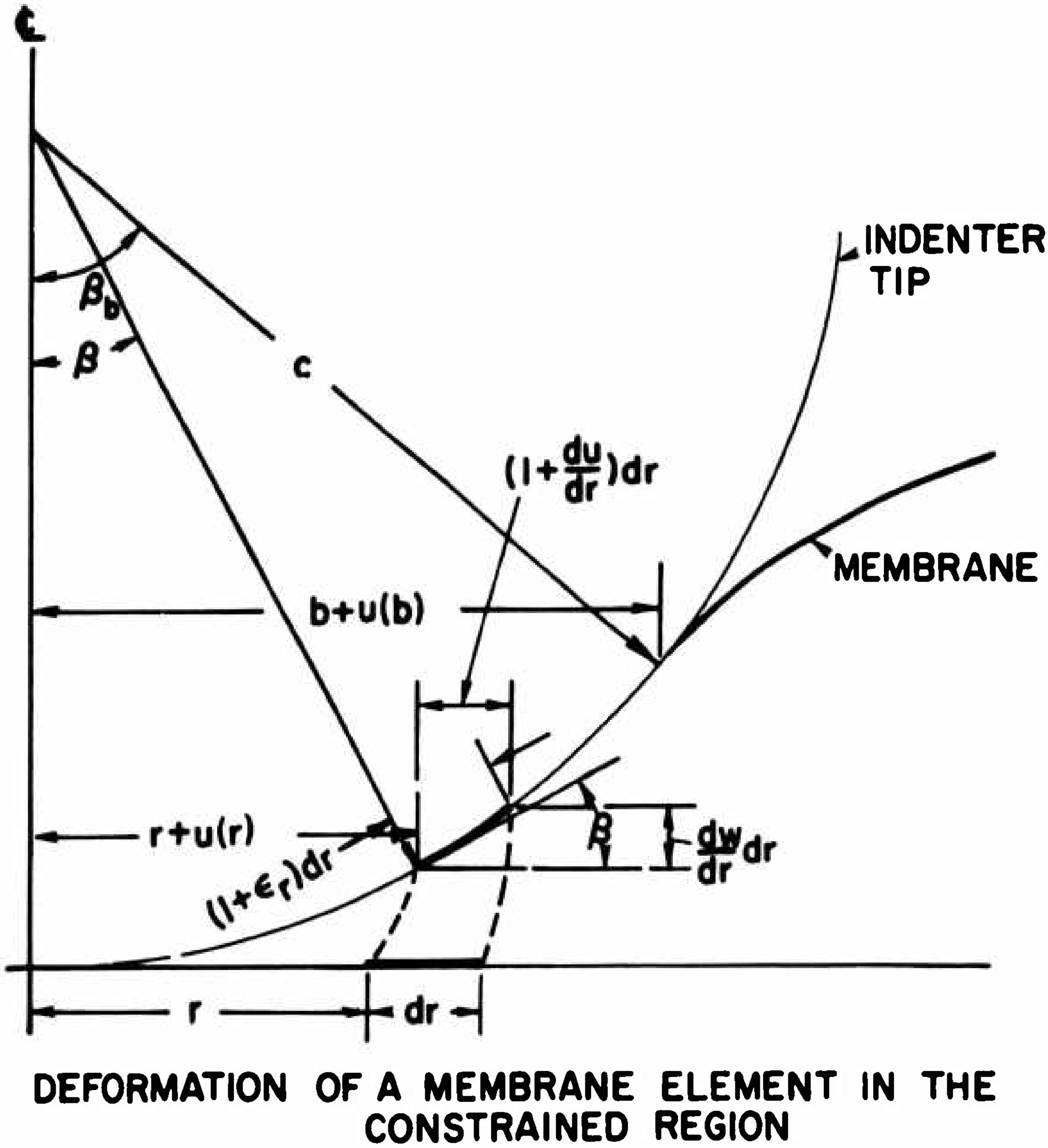


Fig. 2. Deformation of the Elastic Sheet – For the Constrained Region.

plane are w and u, respectively. The angle of rotation of the tangent to the midsurface from the initial plane is β. The sheet has uniform thickness h and elastic coefficients E, ν. Nondimensional stress variables s_r and s_θ are defined as

$$s_r \equiv \frac{\sigma_r}{E} \quad \text{and} \quad s_\theta = \frac{\sigma_\theta}{E} \tag{1.1a}$$

and the independent variable is chosen as

$$y \equiv \left(\frac{r}{a}\right)^2 \tag{1.1b}$$

with use of the nondimensional shape parameter ϵ and the load parameter ρ,

$$\epsilon \equiv \frac{b}{a}, \qquad \rho \equiv \frac{P}{2\pi Ehb}, \tag{1.1c}$$

a nondimensional stress function F is defined as

$$F \equiv (4\epsilon\rho)^{1/3}\, y^{1/2}\, \operatorname{ctn}\beta \cong (4\epsilon\rho)^{1/3}\, y^{1/2}\, \beta^{-1} \tag{1.1d}$$

Stresses and displacements are given in terms of $F(y)$ by Eqs. (3.7), (3.8), (3.9), (3.11) and (3.13) of Ref. 1 as follows:

$$\frac{u}{a} = \left(\frac{1}{2}\epsilon\rho\right)^{2/3} y^{1/2} \left[2\frac{dF}{dy} - (1+\nu)\frac{F}{y}\right] \tag{1.2}$$

$$s_\theta = \left(\frac{1}{2}\epsilon\rho\right)^{2/3} \left(2\frac{dF}{dy} - \frac{F}{y}\right) \tag{1.3}$$

$$s_r = \left(\frac{1}{2}\epsilon\rho\right)^{2/3} \frac{F}{y} \tag{1.4}$$

$$\frac{w}{a} = -\left(\frac{1}{2}\epsilon\rho\right)^{1/3} \int_y^1 \frac{dy}{F} \tag{1.5}$$

$$\beta = (4\epsilon\rho)^{1/3}\, y^{1/2}\, F^{-1} \tag{1.6}$$

Equations (1.3) and (1.4) are seen to satisfy identically the differential equations of equilibrium (Eqs. 2.1a,b below with $p = 0$), and the equation of compatibility [Eq. (2.5b)] becomes, with the use of Eq. (1.6), a differential equation for $F(y)$:

$$\frac{d^2F}{dy^2} + 2F^{-2} = 0 \tag{1.7a}$$

The integral of this corresponding to $s_\theta > 0$ has the general form

$$1 - y = \int_{F(y)}^{F(1)} \left(\frac{1}{v} + C_o\right)^{-(1/2)} dv \ , \tag{1.7b}$$

where C_o is a real-valued constant.

The boundary condition at $y = 1$ is $Ehs_r \equiv h\sigma_{ro} = H_o$. Thus H_o determines the prestress σ_{ro}. The parameter F_o expresses the combined loading condition, through use of Eq. (1.4), as

$$F(1) = F_o \equiv \frac{H_o}{Eh} \left(\frac{4\pi Eha}{P}\right)^{2/3} \tag{1.8a}$$

The boundary condition at $y = \epsilon^2$ is expressed as

$$F(\epsilon^2) = F_\epsilon \tag{1.8b}$$

where F_ϵ is to be determined from conditions of continuity with the constrained region. When $\epsilon, F_o, F_\epsilon$ are given so that $F_o > F_\epsilon > 0$, C_o is uniquely determined from Eq. (1.7b) expressed at $y = \epsilon^2$ as follows:

$$1 - \epsilon^2 = \int_{F_\epsilon}^{F_o} \left(\frac{1}{v} + C_o\right)^{-(1/2)} dv \tag{1.9}$$

SECTION 2. SHEET IN FRICTIONLESS CONTACT WITH INDENTOR

Finite deformation, small strain equations for the constrained region $(0 \le r \le b)$ of the sheet are [Ref. 3] those of equilibrium,

$$\frac{d}{dr}(rs_r \cos\beta) - s_\theta + \frac{rp}{Eh}\sin\beta = 0 \tag{2.1a}$$

$$\frac{d}{dr}(rs_r \sin\beta) - \frac{rp}{Eh}\cos\beta = 0 \tag{2.1b}$$

elasticity $[0 \le \nu \le (1/2)]$,

$$\epsilon_r = s_r - \nu s_\theta \tag{2.2a}$$

$$\epsilon_\theta = s_\theta - \nu s_r \tag{2.2b}$$

and strain-displacement,

$$(1 + \epsilon_r)\sin\beta = \frac{dw}{dr} \tag{2.3a}$$

$$(1 + \epsilon_r)\cos\beta = 1 + \frac{du}{dr} \tag{2.3b}$$

$$\epsilon_\theta = \frac{u}{r} \quad . \tag{2.3c}$$

When p is eliminated between Eqs. (2.1a,b), there obtains

$$\frac{d}{dr}(rs_r) = s_\theta \cos\beta \tag{2.4}$$

The compatibility relation obtained by combining Eqs. (2.3b,c),

$$\frac{d}{dr}(r\epsilon_\theta) + 1 - (1 + \epsilon_r)\cos\beta = 0 \quad , \tag{2.5a}$$

can be written in a more convenient form, with the use of Eqs. (2.2) and (2.4), as

$$r\frac{d}{dr}(s_\theta + s_r) + (1 + s_\theta + s_r)(1 - \cos\beta) = 0 \tag{2.5b}$$

Since the membrane is assumed to deform smoothly onto the indenter, the geometric relation for the constrained region

$$\sin \beta = \frac{r + u(r)}{c} = \frac{r}{c} (1 + \epsilon_\theta) \tag{2.6}$$

is apparent from Fig. 2.

It is next shown that $|\beta|$ must necessarily be of the order of the square root of the maximum elastic strain or smaller. From Eq. (2.3b) and the derivative of both sides of Eq. (2.6),

$$\frac{d\beta}{dr} = c^{-1} (1 + \epsilon_r) \quad , \tag{2.7}$$

transforming as Eqs. (2.6) and (2.7). The differential operator

$$r \frac{d()}{dr} = \frac{r}{c} \frac{d\beta}{d(r/c)} \cdot \frac{d()}{d\beta} = \left(\frac{1 + \epsilon_r}{1 + \epsilon_\theta} \right) \sin \beta \frac{d()}{d\beta} \cdot$$

Equation (2.5b) can be written

$$\frac{d}{d\beta} (s_\theta + s_r) + \left[\frac{(1+s_\theta+s_r)(1+\epsilon_\theta)}{(1 + \epsilon_r)} \right] \frac{(1 - \cos \beta)}{\sin \beta} = 0$$

Because of the assumption of small strains, $|s_r| \ll 1$ and $|s_\theta| \ll 1$, the square bracketed terms in the above equation can be approximated by unity, and the equation becomes

$$\frac{d}{d\beta} (s_r + s_\theta) + \tan \frac{\beta}{2} = 0 \tag{2.8}$$

Let $s(\beta) \equiv s_r + s_\theta$; then the integral of Eq. (2.8) can be written as

$$s(\beta) = s(0) + 2 \ln \cos \frac{\beta}{2} \tag{2.9}$$

or also as

$$\cos \frac{\beta}{2} = \exp \left\{ - \frac{1}{2} [s(0) - s(\beta)] \right\} \tag{2.10}$$

Equation (2.10) shows that $s(\beta) \leq s(0)$ for all $\beta \geq 0$. Furthermore, since $|s|$ is bounded by some linear elastic limit value denoted by

$(2s_L)$, where $s_L \ll 1$, then Eq. (2.10) implies

$$[1 - \cos(\beta/2)] \le 1 - \exp(-s_L) \doteq s_L$$

so that $\beta^2 \lessapprox 8s_L$. Therefore, if s_L is sufficiently small, it is necessary to assume as consistent with the small strain approximation that the rotation is finite but "moderate", viz.

$$\beta^2 \ll 1 \tag{2.11}$$

Hence Eqs. (2.6) and (2 9) are approximated as

$$\sin\beta \doteq \beta \doteq \frac{r}{c} \tag{2.12a}$$

$$s(\beta) \doteq s(0) - \frac{1}{4}\beta^2 \tag{2.12b}$$

If Eq. (2.4) is written in the form

$$r\frac{ds_r}{dr} + 2s_r = s - s_\theta(1 - \cos\beta)$$

then with the use of Eqs. (2.12a,b) and the small strain assumption, this becomes

$$\beta\frac{ds_r}{d\beta} + 2s_r = s(0) - \frac{1}{4}\beta^2 \tag{2.13}$$

The solution is

$$s_r = \frac{C_2}{\beta^2} + \frac{s(0)}{2} - \frac{1}{16}\beta^2 \tag{2.14a}$$

and from Eq. (2.12b),

$$s_\theta = -\frac{C_2}{\beta^2} + \frac{s(0)}{2} - \frac{3}{16}\beta^2 \tag{2.14b}$$

With use of these relations in Eqs. (2.2b) and (2.3c), u is determined to be

$$\frac{u}{c} = -(1 + \nu)\frac{C_2}{\beta} + \frac{1}{2}(1 - \nu)\beta\, s(0) - \frac{1}{16}(3 - \nu)\beta^3 \tag{2.15}$$

The contact pressure p between the membrane and the indenter is determined by using Eqs. (2.1b), (2.12a) and (2.19a,b):

$$p = \frac{Eh}{c}\left[s(0) - \frac{1}{4}\ \beta^2\right] \tag{2.16}$$

An expression can be given for $s(0)$ in terms of $s_r(\beta_b)$, where $\beta_b \equiv (b/c)$, by using Eq. (2.14a):

$$s(0) = 2\ s_r(\beta_b) - 2\ \frac{C_2}{\beta_b^2} + \frac{1}{8}\ \beta_b^2 \tag{2.17}$$

Equations (2.14) to (2.17) are valid for $C_2 \neq 0$, but if the membrane is to be elastic for all $\beta \geq 0$, then $C_2 = 0$. For the case $C_2 = 0$, Eqs. (2.14) to (2.16) become with use of Eq. (2.17):

$$s_r(\beta) = s_r(\beta_b) + \frac{1}{16}\ (\beta_b^2 - \beta^2) \tag{2.18a}$$

$$s_\theta(\beta) = s_r(\beta_b) + \frac{1}{16}\ \beta_b^2 - \frac{3}{16}\ \beta^2 \tag{2.18b}$$

$$\frac{u(\beta)}{c} = \beta\left[(1 - \nu)\ s_r(\beta_b) + \frac{(1 - \nu)}{16}\ \beta_b^2 - \frac{(3 - \nu)}{16}\ \beta^2\right] \tag{2.18c}$$

$$p(\beta) = \frac{Eh}{c}\left[2\ s_r(\beta_b) + \frac{1}{8}\ \beta_b^2 - \frac{1}{4}\ \beta^2\right] \tag{2.18d}$$

The displacement $w(\beta)$ is found by integration of Eq. (2.3a):

$$w(\beta) = w(\beta_b) - \frac{c}{2}\ (\beta_b^2 - \beta^2) \tag{2.18e}$$

A necessary condition to be satisfied for contact between indenter and membrane in $0 \leq \beta \leq \beta_b$, is that $p(\beta) \geq 0$. This condition is seen to be fulfilled if

$$S_\theta(\beta_b) > 0 \tag{2.19}$$

This latter condition is necessary, however, for the stability of the free membrane against wrinkling [Ref. 1, Eq. (3.16)] and must be imposed for a rotationally symmetric solution to the indenter problem. It will be shown in the following section [see Eq. (3.39)] that (2.19) is satisfied.

The equilibrium of the indenter at the point of tangency, $\beta = \beta_b$, requires

$$P = 2\pi Eh\ b\beta_b\ S_r(\beta_b)$$

which is written as

$$\frac{P}{2\pi Eha} = \beta_b^2\ \frac{c}{a}\ s_r(\beta_b) \qquad (2.20)$$

SECTION 3. SOLUTION OF THE INDENTER PROBLEM

Solutions from Section 1 for the free region and from Section 2 for the constrained region are now used. Solutions for these two regions are distinguished by the use of y as independent variable in the free region and of β as independent variable in the constrained region. At the point of tangency $r = b$ ($y = \epsilon^2$ for the free region and $\beta = \beta_b$ for the constrained region), conditions of continuity of displacements and normal stress are imposed. From Eq. (1.4),

$$s_r(\epsilon^2) = \left(\frac{1}{2}\,\epsilon\rho\right)^{2/3} \frac{F(\epsilon^2)}{\epsilon^2} \tag{3.1}$$

where, from Eq. (1.1c),

$$\epsilon\rho \equiv \frac{P}{2\pi Eha} \tag{3.2}$$

Continuity of s_r requires

$$s_r(\epsilon^2) = s_r(\beta_b) \tag{3.3}$$

This equation, after substitution from Eqs. (1.8b), (2.20), (3.1) and (3.2), can be written as

$$F_\epsilon = \left(\frac{2P}{\pi Eha}\right)^{1/3} \frac{c}{a} \tag{3.4}$$

Note that F_ϵ does not depend upon ϵ .

Continuity of the displacement u is invoked to determine b, or what is equivalent, the nondimensional parameter ϵ:

$$u(\epsilon^2) = u(\beta_b) \tag{3.5}$$

Equations (1.2), (1.7a), (1.8b) and (3.2) give

$$\frac{u(\epsilon^2)}{b} = \left(\frac{P}{4\pi Eha}\right)^{2/3} \left[2\left(\frac{1}{F_\epsilon} + c_o\right)^{1/2} - (1 + \nu)\,\frac{F_\epsilon}{\epsilon^2}\right] \tag{3.6}$$

Equation (2.18c) gives

$$\frac{u(\beta_b)}{b} = (1 - \nu)\, S_r(\beta_b) - \frac{1}{8}\beta_b^2 \tag{3.7a}$$

which can be written in the convenient form, using Eqs. (3.1) to (3.4),

$$\frac{u(\beta_b)}{b} = \left(\frac{P}{4\pi Eha}\right)^{2/3} \left[(1 - \nu)\frac{F_\epsilon}{\epsilon^2} - \frac{\epsilon^2}{2F_\epsilon^2}\right] \tag{3.7b}$$

Substitution from Eqs. (3.7b) and (3.6) into Eq. (3.5) leads to the following equation quadratic in ϵ^2,

$$\epsilon^4 + 4F_\epsilon^2\left(\frac{1}{F_\epsilon} + C_o\right)^{1/2} \epsilon^2 - 4F_\epsilon^3 = 0 \tag{3.8}$$

The only positive root is

$$\epsilon^2 = -2\, F_\epsilon^2\left(\frac{1}{F_\epsilon} + C_o\right)^{1/2} + \left[4F_\epsilon^4\left(\frac{1}{F_\epsilon} + C_o\right) + 4F_\epsilon^3\right]^{1/2}$$

which can be written simply as

$$\epsilon^2 = 2F_\epsilon^2\left[\left(\frac{2}{F_\epsilon} + C_o\right)^{1/2} - \left(\frac{1}{F_\epsilon} + C_o\right)^{1/2}\right] \tag{3.9}$$

Equations (1.9) and (3.9) are used to determine ϵ and C_o with F_o and F_ϵ, or equivalently, with P and H_o, through the use of Eqs. (1.8a) and (3.4), considered as independent variables.

Certain characteristics of the solutions ϵ and C_o to Eqs. (1.9) and (3.9) for fixed $H_o > 0$ are now studied. Restrictions of these developments to small indenter radii will be made subsequently as sufficient for applied purposes and in the interest of brevity.

First, it is observed that F_ϵ is expressible as a function of F_o and H_o by means of Eqs. (1.8a) and (3.9)

$$F_\epsilon = 2\left(\frac{H_o}{Eh}\right)^{1/2} \frac{c}{a}\, F_o^{-(1/2)} \tag{3.10}$$

Therefore, for fixed H_o, F_ϵ may be regarded as a function of F_o only, viz. $F_\epsilon(F_o)$. Substitution of $F_\epsilon(F_o)$ into Eq. (3.9) yields

the function $\epsilon = \epsilon(F_o, C_o)$. Then the right-hand side of Eq. (1.9) can be defined by the function $I(F_o, C_o)$ as

$$I(F_o, C_o) = \int_{F_\epsilon(F_o)}^{F_o} \left(\frac{1}{v} + C_o\right)^{-(1/2)} dv \tag{3.11}$$

and then Eq. (1.9) can be written as

$$\epsilon^2(F_o, C_o) + I(F_o, C_o) - 1 = 0 \tag{3.12}$$

Equation (3.12) will be shown to determine implicitly a function $C_o(F_o)$, which can be regarded as the solution to the problem. Since $0 \leq \epsilon < 1$ is a geometrical constraint, then by Eq. (3.12) $0 < I[F_o, C_o(F_o)] \leq 1$. It is necessary to consider the left-hand side of Eq. (3.12) in the F_o, C_o plane within the allowable domain*) which is determined by the two inequalities

$$\frac{1}{F_o} + C_o > 0 \tag{3.13a}$$

$$0 < F_\epsilon < \left(\frac{H_o}{Eh}\right)^{1/3} \left(\frac{2c}{a}\right)^{2/3} < F_o \tag{3.13b}$$

Inequality (3.13b) follows from Eq. (3.10) and the inequality $I > 0$, which implies $F_o > F_\epsilon > 0$.

It is evident from Eqs. (3.9), (3.10) and (3.11) that $\epsilon(F_o, C_o)$ and $I(F_o, C_o)$ are continuous functions of F_o and C_o in the allowable domain. The partial derivative with respect to C_o of the left-hand side of Eq. (3.12) is

---

*) It is shown in Ref. 1 that if wrinkling is to be avoided, $C_o(F_o)$ must be confined to a smaller domain which lies wholly within the allowable domain.

$$\frac{\partial}{\partial C_o} (\epsilon^2 + I - 1) = -\frac{1}{2}\epsilon^2\left(\frac{2}{F_\epsilon} + C_o\right)^{-(1/2)}\left(\frac{1}{F_\epsilon} + C_o\right)^{-(1/2)}$$

$$- \int_{F_\epsilon}^{F_o} \left(\frac{1}{v} + C_o\right)^{-3/2} dv \qquad (3.14a)$$

Therefore, in the allowable domain, the right-hand side above is negative, and so

$$\frac{\partial}{\partial C_o} (\epsilon^2 + I - 1) < 0 \qquad (3.14b)$$

This partial derivative is a continuous function of F_o and C_o.

At this point, the further restriction to small indenter radii is made, viz.

$$\frac{c}{a} << 1 \qquad (3.15a)$$

From this it necessarily follows that

$$\epsilon << 1 \qquad (3.15b)$$

since $\sigma_{ro} \leqq \sigma_L$ and

$$s_L \equiv (\sigma_L/E) << 1 \qquad (3.15c)$$

then $E^{-1} \sigma_{ro} << 1$, and therefore

$$(H_o/Eh) << 1 \qquad (3.15d)$$

Consequently, it can easily be shown from Eq. (3.10) and inequalities (3.13b), (3.15a,d) that

$$F_\epsilon < 2^{2/3} \left(\frac{H_o}{Eh}\right)^{1/3} \left(\frac{c}{a}\right)^{2/3} << 1 \ . \qquad (3.15e)$$

When $C_o = 0$, let $F_o = F_o^*$ be the root of Eq. (3.12). Then with the use of Eqs. (3.9) and (3.11), Eq. (3.12) yields

$$F_o^{*(3/2)} = \frac{3}{2}\left[1 - 2\left(\sqrt{2} - \frac{4}{3}\right) F_\epsilon^{*(3/2)}\right] \tag{3.16a}$$

In view of (3.15e)

$$F_o^* \cong \left(\frac{3}{2}\right)^{2/3} + O(\epsilon^2) \tag{3.16b}$$

It follows from inequalities (3.13), (3.15a,d) that the point $F_o = F_o^*$ $C_o = 0$, lies within the allowable domain.

With these restrictions, we will now prove the existence of C_o as a continuous function of F_o in the following subdomain of the allowable domain:

$$C_o \geq 0 \quad \text{and} \quad F_o \geq F_o^* \tag{3.17}$$

Inequality (3.14b) implies that $(\epsilon^2 + I - 1)$ is a monotone, strictly decreasing function of C_o for a fixed F_o. Also for $C_o = 0$ and $F_o > F_o^*$, it can readily be shown that

$$\epsilon^2 + I - 1 > 0 \tag{3.18a}$$

It can also be shown, with the use of Eqs. (3.9) and (3.11), that

$$\lim_{\substack{C_o \to \infty \\ F_o \text{ fixed}}} \epsilon^2 = 0 \quad \text{and} \quad \lim_{\substack{C_o \to \infty \\ F_o \text{ fixed}}} I = 0 \ ,$$

from which it follows that

$$\lim_{\substack{C_o \to \infty \\ F_o \text{ fixed}}} (\epsilon^2 + I - 1) = -1 \tag{3.18b}$$

Therefore, for each $F_o \geq F_o^*$, it follows from (3.14b), (3.18a,b) that there exists a unique positive value of C_o, such that Eq. (3.12) is

satisfied. The continuity of $C_o(F_o)$ follows from a theorem on implicit functions (Ref. 4).

In the subdomain (3.17), upper and lower bounds for $C_o(F_o)$ are obtained when Eq. (3.11) is rewritten as,

$$I(F_o, C_o) = \int_{F_\epsilon(F_o)}^{F_o} C_o^{-(1/2)}\, dv - C_o^{-(1/2)} \left\{ \int_{F_\epsilon}^{F_o} \left[1 - \left(1 + \frac{1}{vC_o}\right)^{-(1/2)} \right] dv \right\}$$

This becomes

$$I(F_o, C_o) = \frac{F_o - F_\epsilon}{C_o^{1/2}} - C_o^{-(1/2)} R \tag{3.19a}$$

Where

$$R \equiv \int_{F_\epsilon}^{F_o} \left[1 - \left(1 + \frac{1}{vC_o}\right)^{-(1/2)} \right] dv \tag{2.19b}$$

It is easily shown that

$$0 < R < C_o^{-1} \quad \ell n \left(\frac{1 + C_o F_o}{1 + C_o F_\epsilon} \right) \tag{3.19c}$$

Then Eqs. (3.12) and (3.16a) are used to obtain

$$(1 - \epsilon^2) C_o^{1/2} = F_o - F_\epsilon - R < F_o ,$$

which yields

$$F_o^2 \left[1 - \frac{F_\epsilon}{F_o} - \frac{1}{F_o C_o} \quad \ell n \left(\frac{1 + F_o C_o}{1 + F_\epsilon C_o} \right) \right]^2 < C_o < F_o^2 [1 + 0(\epsilon^2)] \tag{3.20a}$$

With the right inequality and Eq. (3.10), an upper bound for $F_\epsilon C_o$ is obtained

$$F_\epsilon C_o < 2 \left(\frac{H_o}{Eh} \right)^{1/2} \frac{c}{a} F_o^{3/2} [1 + 0(\epsilon^2)] \tag{3.20b}$$

The partial differential of the expression in Eq. (3.12) with respect to F_o is

$$\frac{\partial}{\partial F_o}(\epsilon^2 + I - 1) = \left(\frac{1}{F_o} + C_o\right)^{-(1/2)} + \frac{\partial F_\epsilon}{\partial F_o}\left[\frac{2\epsilon^2}{F_\epsilon} - 2\left(\frac{2}{F_\epsilon} + C_o\right)^{-(1/2)}\right] \tag{3.21a}$$

where with the use of Eq. (3.10),

$$\frac{\partial F_\epsilon}{\partial F_o} = -\frac{1}{2} F_\epsilon F_o^{-1} \tag{3.21b}$$

Then Eq. (3.22a) becomes

$$\frac{\partial}{\partial F_o}(\epsilon^2 + I - 1) = \left(\frac{1}{F_o} + C_o\right)^{-(1/2)} - \frac{\epsilon^2}{F_o} + \frac{F_\epsilon}{F_o}\left(\frac{2}{F_\epsilon} + C_o\right)^{-(1/2)} \tag{3.21c}$$

This with the use of the left inequality (3.20) yields

$$\frac{\partial}{\partial F_o}(\epsilon^2 + I - 1) > \frac{1}{F_o}\left\{\left[1 + O(\epsilon^2) + \frac{1}{F_o^3}\right]^{-(1/2)} - \epsilon^2 + F_\epsilon\left(\frac{2}{F_\epsilon} + C_o\right)^{-(1/2)}\right\} \tag{3.22a}$$

From this inequality it is evident that

$$\frac{\partial}{\partial F_o}(\epsilon^2 + I - 1) > 0 \tag{3.22b}$$

in the subdomain (3.17).

The differential of Eq. (3.12) yields an expression for the derivative of $C_o(F_o)$:

$$\frac{dC_o}{dF_o} = -\left[\frac{\partial}{\partial F_o}(\epsilon^2 + I - 1) \Big/ \frac{\partial}{\partial C_o}(\epsilon^2 + I - 1)\right] , \tag{3.23}$$

In view of inequalities (3.14b) and (3.22b), this derivative is positive for all $F_o \geq F_o^*$. This implies that the solution to Eq. (3.12) expressed by $C_o(F_o)$ is a continuous, monotone strictly increasing function.

Approximate solutions for the indenter problem will next be obtained for certain limiting cases. Consider first the case when

$$P = P_L \equiv 4\pi Ehc\, s_L^2 \tag{3.25a}$$

and

$$\left(\frac{H_o}{Eh}\right) > \left(\frac{c}{a}\right)^{2/3} s_L^{4/3}\, F_o^* \tag{3.25b}$$

Then, with the use of Eq. (1.8a)

$$F_o\Big|_{P=P_L} = F_{oL} \equiv \left(\frac{H_o}{Eh}\right)\left(\frac{4\pi Eha}{P_L}\right)^{2/3} \equiv \left(\frac{H_o}{Eh}\right)\left(\frac{a}{c}\right)^{2/3} \frac{1}{s_L^{4/3}} \tag{3.25c}$$

It can be seen with the use of inequality (3.25b), that $F_{oL} > F_o^*$, and therefore the solution for $P = P_L$ is in the subdomain (3.17). Upon substitution of $F_o = F_{oL}$ into the right-hand inequality (3.20b), there is obtained for $F_\epsilon = F_{\epsilon L}$ and $C_o = C_{oL}$,

$$F_\epsilon C_o = F_{\epsilon L} C_{oL} < 2\left(\frac{\sigma_{ro}}{\sigma_L}\right)^2 \left[1 + 0(\epsilon^2)\right] \tag{3.25d}$$

Therefore, if the ratio (σ_{ro}/σ_L) obeys

$$2 s_L^{2/3}\left(\frac{c}{a}\right)^{4/3} F_o^{*2} < 2\left(\frac{\sigma_{ro}}{\sigma_L}\right)^2 \ll 1 \quad , \tag{3.26a}$$

where the left inequality is obtained from Eq. (3.25b), then it follows that for $F_o = F_{oL}$

$$F_\epsilon C_o = F_{\epsilon L} C_{oL} \ll 1 \tag{3.26b}$$

Hence the following order relation will hold, by continuity, for F_o in some neighborhood of F_{oL} and for P in some neighborhood of P_L:

$$0 < F_\epsilon C_o \ll 1 \tag{3.26c}$$

Let Eq. (3.9) be approximated for small $F_\epsilon C_o$ as

$$\epsilon = [2(\sqrt{2} - 1)]^{1/2} \; F_\epsilon^{3/4} \; [1 - O(F_\epsilon C_o)] \tag{3.27a}$$

$$\cong 2(\sqrt{2} - 1)^{1/2} \; \frac{c}{a}^{3/4} \left(\frac{P}{2\pi Eha}\right)^{1/4} \tag{3.27b}$$

Hence

$$b \cong 2(\sqrt{2} - 1)^{1/2} \left(\frac{Pc^3}{2\pi Eh}\right)^{1/4} \tag{3.27c}$$

Expressions for β_b , $s_r(\beta_b)$ and $s_r(0)$ for P near P_L are obtained from Eqs. (2.18a) and (2.19) with the use of the approximation (3.27c):

$$\beta_b = 2(\sqrt{2} - 1)^{1/2} \left(\frac{P}{2\pi EhC}\right)^{1/4} \tag{3.28a}$$

$$s_r(\beta_b) = \frac{1}{4(\sqrt{2} - 1)} \left(\frac{P}{2\pi Ehc}\right)^{1/2} = \frac{1}{16(\sqrt{2} - 1)} \beta_b^2 \tag{3.28b}$$

$$s_r(0) = \left(1 + \frac{1}{(\sqrt{2} - 1)^2}\right) \frac{1}{16} \beta_b^2 \tag{3.28c}$$

It is noted that these relations are independent of the prestress and membrane radius a. Also the maximum stress in the membrane, which occurs at $\beta = 0$(i.e. $r = 0$), increases with P. Then for $P = P_L$, the maximum stress in the membrane, evaluated with the use of Eqs. (3.28a,c) and (3.25a), is

$$s_r(0)\Big|_{P=P_L} = s_L \; , \tag{3.29}$$

which is the elastic limit or yield stress σ_L . Therefore, since the stresses are limited by σ_L , it is concluded that if the prestress is within the limit given by (3.26a), then P_L represents an upper bound on P , viz.

$$P \leq P_L \; , \tag{3.30}$$

If P is sufficiently small, so that

$$F_o \gg 1 \,, \tag{3.31a}$$

then it is easily shown with the use of inequality (3.20a) that

$$C_o \cong F_o^2 \tag{3.31b}$$

Sufficiently large values of F_o are now considered, such that

$$F_\epsilon C_o \gg 1 \quad \text{and} \quad F_\epsilon F_o^2 \gg 1 \,. \tag{3.31c}$$

This inequality, with the use of Eqs. (1.8a) and (3.4), implies that

$$\frac{E}{8\pi hc} \frac{P}{(H_o/h)^2} \ll 1 \tag{3.31d}$$

With the use of inequality (3.31c), Eq. (3.9) can be approximated as

$$\epsilon^2 \cong \frac{F_\epsilon}{C_o^{1/2}} \left[1 - \frac{3}{4F_\epsilon C_o} \right] \tag{3.32a}$$

This expression with the use of Eqs. (3.31b), (1.8a) and (3.10) yields

$$\epsilon \cong \left(\frac{F_\epsilon}{F_o}\right)^{1/2} = \left[\frac{c}{2\pi a^2}\left(\frac{P}{H_o}\right)\right]^{1/2} \tag{3.32b}$$

An approximation for the integral $I(F_o, C_o)$ in Eq. (3.11) is also obtained with the use of inequalities (3.31a,c) as

$$I(F_o, C_o) \cong \frac{F_o}{C_o^{1/2}} \left(1 - \frac{F_\epsilon}{F_o} - \frac{1}{2F_o C_o} \ell n \frac{F_o}{F_\epsilon}\right) \tag{3.33}$$

An expression for deflection $w(\epsilon^2)$ is obtained from Eqs. (1.5) and (1.7a);

$$\frac{w(\epsilon^2)}{r} = -2\left(\frac{H_o/Eh}{F_o}\right)^{1/2} \left[F_o\left(\frac{1}{F_o} + C_o\right)^{1/2} - F_\epsilon\left(\frac{1}{F_\epsilon} + C_o\right)^{1/2} - C_o(1 - \epsilon^2) \right] \tag{3.34a}$$

With the use of Eqs. (3.32a) and (3.33) and inequalities (3.31a,c), this equation is approximated as

$$\frac{w(\epsilon^2)}{a} \doteq -\left(\frac{H_o}{Eh}\right)^{1/2} \frac{1}{(F_o C_o)^{1/2}} \ln\left(\frac{F_o}{F_\epsilon}\right) \tag{3.34b}$$

With the use of Eqs. (3.31b), (1.8a) and (3.10), Eq. (3.34b) yields

$$w(\epsilon^2) = -\frac{P}{4\pi H_o} \ln\left[\frac{2\pi a^2}{c}\left(\frac{H_o}{P}\right)\right] \tag{3.34c}$$

The central deflection $w(\beta = 0)$ is obtained from Eq. (2.18e) as

$$w(0) = -\frac{c}{2}\beta_b^2 + w(\epsilon^2) \tag{3.35a}$$

This yeilds, with the use of Eq. (3.34c) and (3.32b)

$$w(0) = -\frac{P}{4\pi H_o}\left[1 + \ln\left(\frac{2\pi a^2}{c}\,\frac{H_o}{P}\right)\right] \tag{3.35b}$$

Expressions for $s_r(\beta_b)$ and $s_r(0)$ are obtained with the use of approximation (3.32b) and Eqs. (2.18a) and (2.20) :

$$s_r(\beta_b) = \frac{H_o}{Eh} \tag{3.36a}$$

$$s_r(0) = \frac{H_o}{Eh} + \frac{1}{32\pi c}\,\frac{P}{H_o} \tag{3.36b}$$

The final topic considered in this section is the stability of the free membrane against wrinkling. It has been shown in Ref. 1 (see Appendix; II, p. 43), that the wrinkling stability condition is satisfied if

$$s_\theta(1) \geq 0 \tag{3.37a}$$

and

$$s_\theta(\epsilon^2) \geq 0 \tag{3.37b}$$

It can easily be shown that (3.37a) is satisfied if

$$C_o \geq \frac{F_o^2}{4} - \frac{1}{F_o} \tag{3.38}$$

This inequality is satisfied both for $F_o = F_o^*$ and for $F_o >> 1$. Furthermore the numerical solutions for $C_o(F_o)$ show that inequality (3.38a) is satisfied for all $F_o \geq F_o^*$.

An expression for $s_\theta(\epsilon^2)$ is obtained with the use of Eqs. (2.18b) and (2.20) as

$$s_\theta(\epsilon^2) = \frac{1}{8\left(\frac{c}{a}\right)^2 \epsilon^2} \left[\frac{4P}{\pi Ehc}\left(\frac{c}{a}\right)^4 - \epsilon^4\right] \tag{3.39}$$

With the use of Eq. (3.9), it can be shown that for both $0 < F_\epsilon C_o < 1$ and $2 < F_\epsilon C_o < \infty$,

$$\epsilon^2 < 2(\sqrt{2} - 1)F_\epsilon^{3/2} \tag{3.40a}$$

Furthermore from numerical solutions this inequality has been observed to be satisfied for $1 < F_\epsilon C_o < 2$. Inequality (3.40a), with the use of Eq. (3.4), implies that

$$\epsilon^2 < \left(\frac{4P}{\pi Ehc}\right)^{1/2} \left(\frac{c}{a}\right)^2 \tag{3.40b}$$

Therefore, with the use of Eqs. (3.39) and (3.40b), it is concluded that inequality (3.37b) is satisfied for all $F_o \geq F_o^*$. Hence, in particular, the membrane is stable against wrinkles for $P \leq P_\ell$.

SECTION 4. NUMERICAL RESULTS AND COMPARISON WITH EXPERIMENTS

Experimental results have been reported (Ref. 2) for the central load-transverse deflection characteristics and the deflection profile of a mylar sheet*) stretched in its plane by dead weight loading. This load is referred to as the platen load. The sheet was supported by a ring of 5-in inner radius. The transverse load P was applied at the center of the sheet by a load probe, or indenter, having a hemispherical tip of 1/16-in radius. The membrane deflections were measured both by a dial gage and, in the immediate vicinity of the indenter, by measurements from photographic enlargements. Stresses were not measured directly, but σ_r was calculated from an equilibrium equation [the present Eq. (2.1b) with $p = 0$], and values of β were obtained by numerical differentiation of the deflections.

The procedure used to compute numerical results from the present theory is first to obtain, for given values of P and σ_{ro} (i.e., for a given F_o), the root C_o of Eq. (3.12). Since for these experiments, $(c/a) = 0.0125$, inequalities (3.15a,b) follow. Then, for each $F_o > F_o^*$ [Eq. (3.16)], the root $C_o > 0$ is obtained using numerical integration to evaluate $I(F_o,C_o)$, Eq. (3.11). Then $\epsilon = \epsilon(F_o,C_o)$, follows immediately. With the use of these values of ϵ and C_o, and with application of Eqs. (1.7b), (1.4), (1.3), (1.2) and (1.5), stresses σ_r,σ_θ and deflections w,u in the free region of the membrane are calculated. Stresses and deflections in the constrained region of the membrane are then determined with the use of Eqs. (3.3), (2.18) and (2.20); note that $\beta_b = (a/c)\epsilon$, and that $w(\epsilon^2)$ from Eq. (1.5) is equal to $w(\beta_b)$ in Eq. (2.18e).

The values of σ_{ro} could not be obtained experimentally to within less than 15% error (see Ref. 2, Table A1), probably because of friction between the sheet and the outer supports. Even for constant platen

*)Properties of mylar sheet: Thickness $h = 6.0 \times 10^5$ psi Poisson's ratio $\nu = 0.3$, Yield stress in uniaxial tension $\sigma_y \cong 10^4$ psi.

loads in the experiment, σ_{ro} is a weak function of P. To compare predictions of the present theory with the experiments, the numerical value of σ_{ro} for each value of P is determined by adjusting σ_{ro} in the theory so that the predicted deflections near the outer edge $r = a$ are in agreement with the experimental deflections. The values of σ_{ro} thus found agree roughly with the experimental estimates.

In Figs. 3 and 4 are shown curves for the principal stresses and the transverse deflection w as predicted by theory for $P = 0.66$ lbs (300 grams) and a calculated value of $\sigma_{ro} = 850$ psi; for these values, $F_o = 1.44$, $P_L = 0.0704$ lbs, and $F_{oL} = 6.4$. Also $(\sigma_{ro}/\sigma_L)^2 = 7.2 \times 10^{-3}$; hence $P = P_L$ at the yeild limit, in view of inequality (3.25c) and Eq. (3.29). Therefore, since $P > P_L$ and $F_o < F_{oL}$, the stresses in the neighborhood of the indenter as predicted by the theory, exceed the yield limit of the material (See Figs. 3 and 4). This indicates that plastic deformation of the membrane in the neighborhood of the indenter must have occurred in the experiment causing larger transverse displacements w than those predicted by the theory.

In Fig. 5, central load-deflection characteristics are shown in curves 1, 2, 3 — each for a fixed prestress. The values of σ_{ro} used are those determined as mentioned above for platen loads of 10,20 and 40 lbs. with the respective indenter loads of 300, 350 and 400 gms. Limiting value of load $P_L = 0.0704$ lb. is also indicated. Since experimental data was not available for loads below P_L, no direct comparison between theory and experiments was possible. However, discrepancy between theory and experiment decreases for smaller loads. In the neighborhood of limit load P_L, theory and experiments show good agreement.

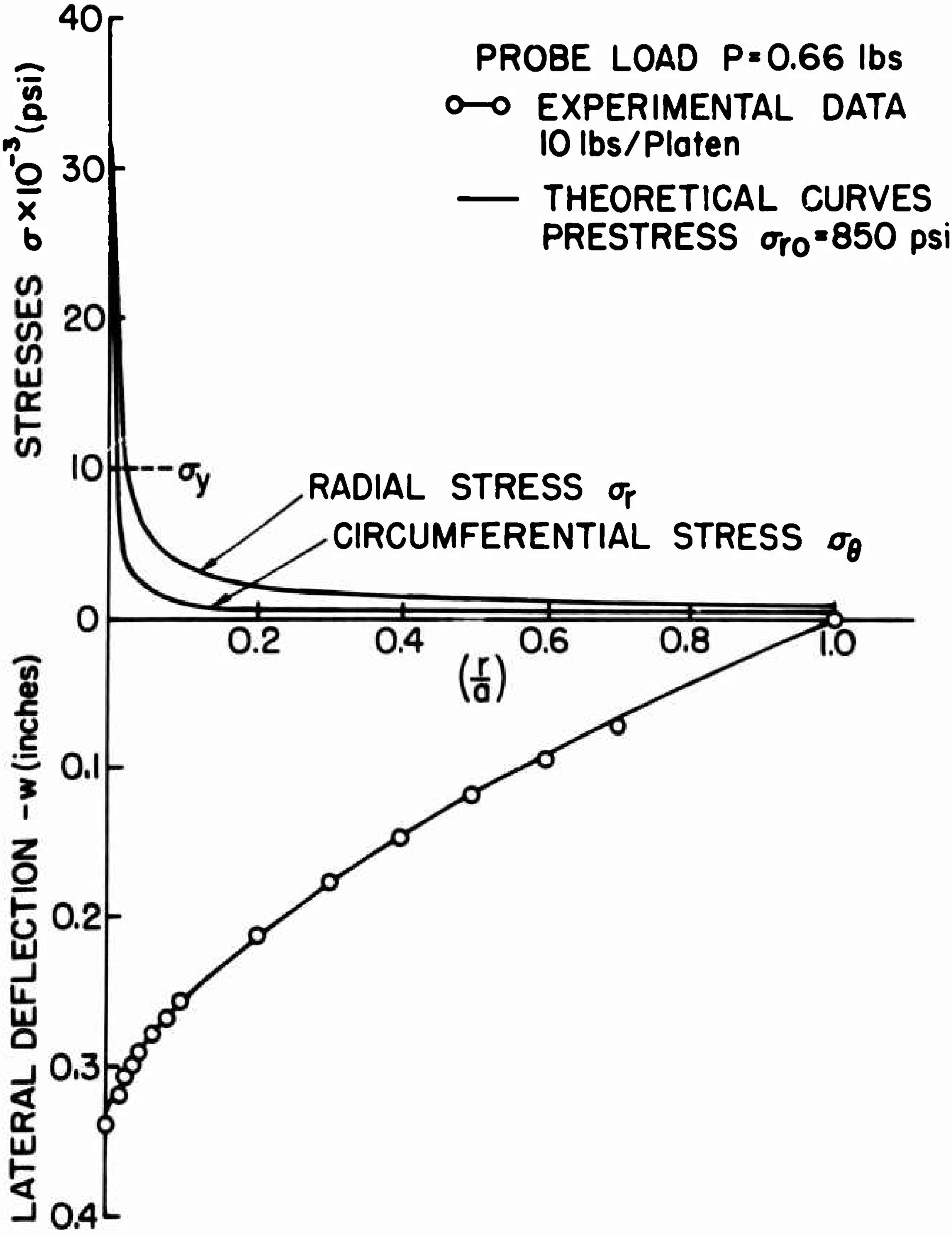


Fig. 3. Deflection w and Stresses σ_r and σ_θ vs. (r/a) for the Membrane.

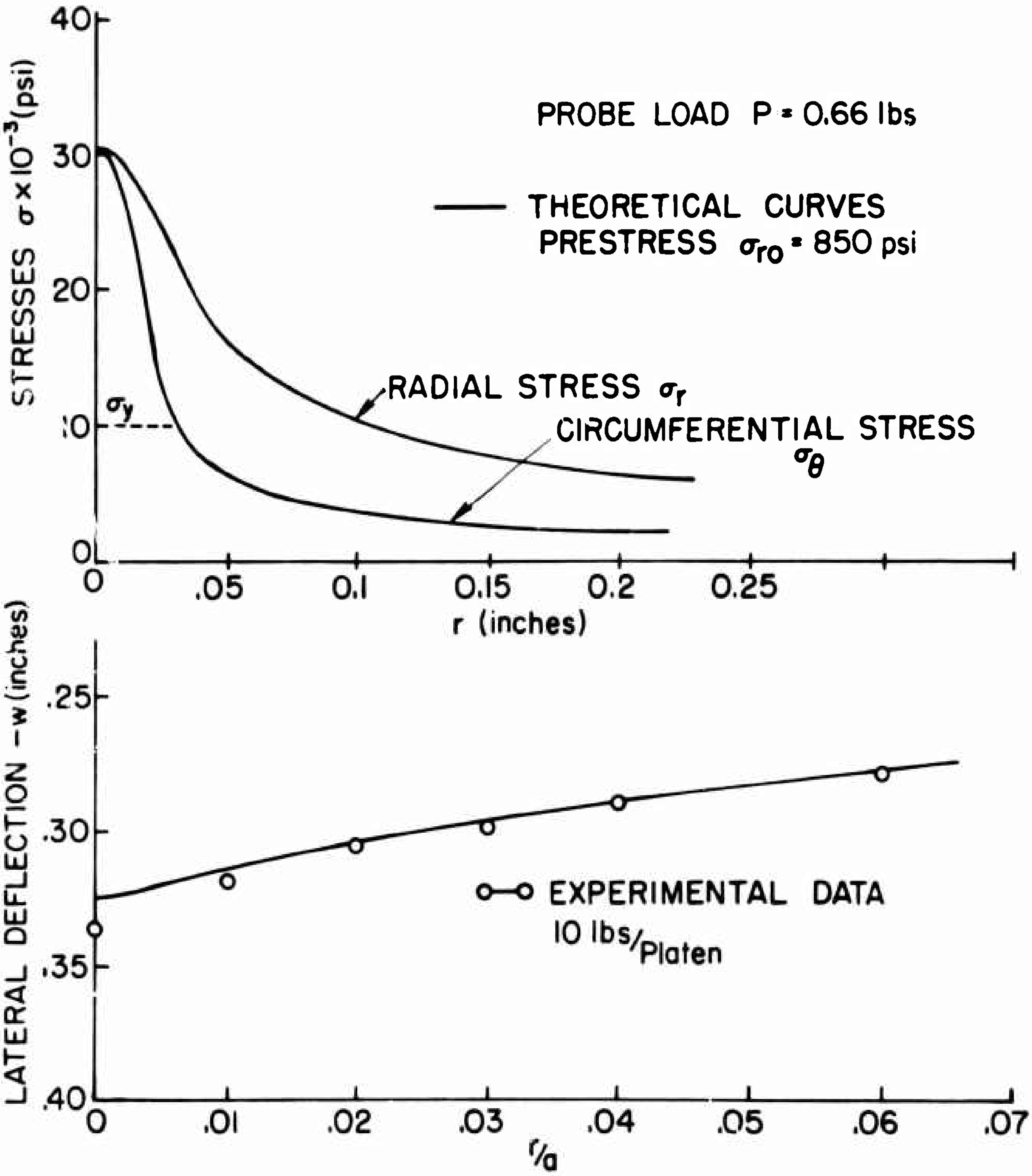


Fig. 4. Deflection w and Stresses σ_r and σ_θ vs. (r/a) for the Membrane in the Neighborhood of the Indenter.

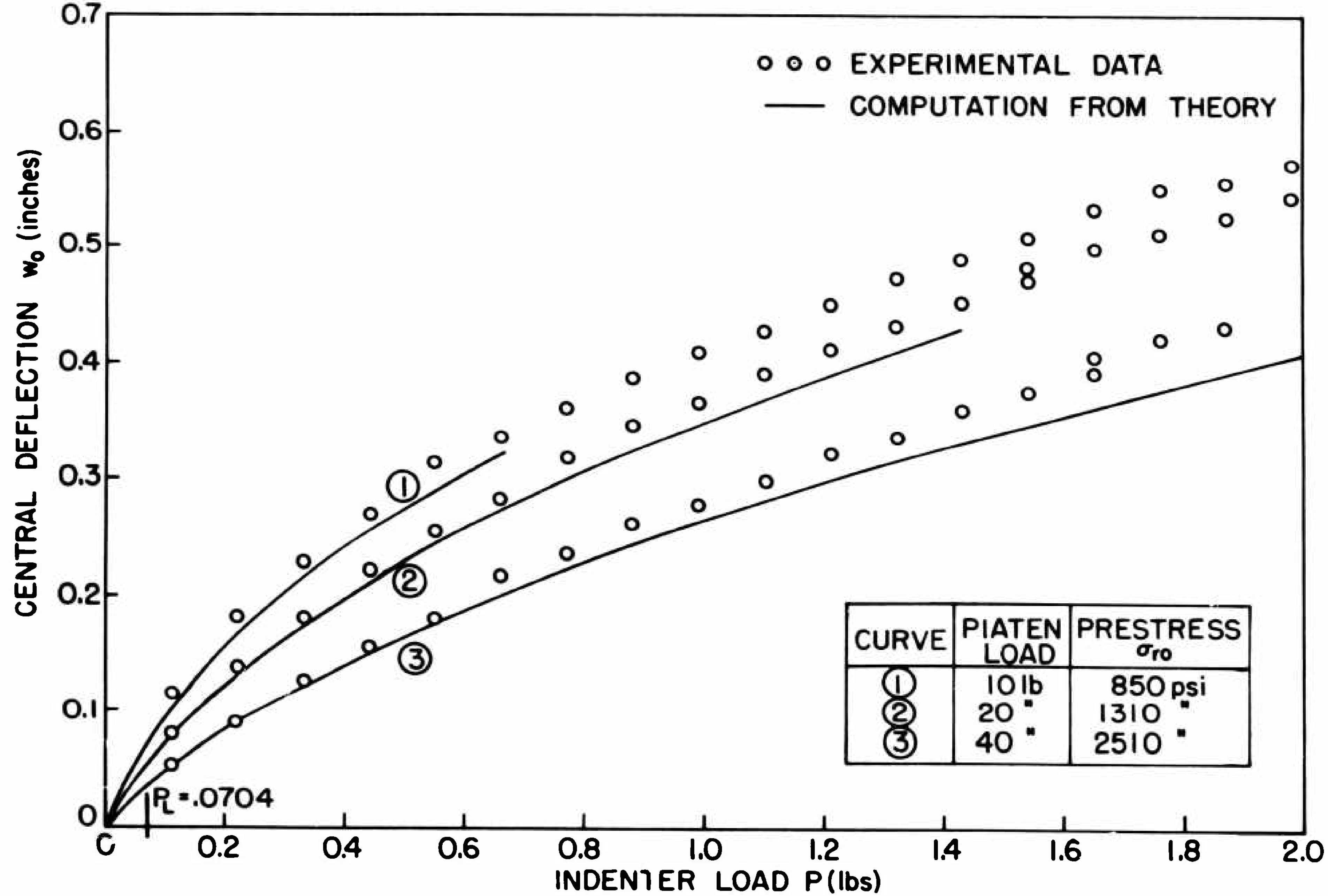


Fig. 5. Central Deflection vs. Indenter Load P for Fixed H_o.

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